Int J Theor Phys (2009) 48: 3358–3364
DOI 10.1007/s10773-009-0139-2

Necessary and Sufficient Condition for Greenberger-Horne-Zeilinger Diagonal States to Be Full N-partite Entangled

Koji Nagata

Received: 7 May 2009 / Accepted: 2 September 2009 / Published online: 16 September 2009
© Springer Science+Business Media, LLC 2009

Abstract We show that any N-qubit state which is diagonal in the Greenberger-Horne-Zeilinger basis is full N-qubit entangled state if and only if no partial transpose of the multiqubit state is positive with respect to any partition.

Keywords Entanglement · Quantum information theory

1 Introduction

Quantum information theory [1, 2] relies on utilizing entangled state. Also there is much research of nature of entangled states related to local realistic theories [3–5]. Separable state and entangled state were defined in 1989 [6]. And it was discussed very much which state is separable or entangled [7–14].

Peres and Horodecki provided the method of classification of a state. A multipartite state ρ has positive partial transposes with respect to all subsystems if ρ is separable state [15]. The partial transpose of an operator on a Hilbert space $H_1 \otimes H_2$ is defined by

$$\left(\sum_l A_l^1 \otimes A_l^2\right)^{T_1} = \sum_l {A_l^1}^T \otimes A_l^2, \tag{1}$$

where the superscript T denotes transposition in the given basis. Especially, Horodecki et al. showed [16] that the condition is sufficient for quantum states to be separable on $H = C^2 \otimes C^2$ and $H = C^2 \otimes C^3$. However it was shown [17] that the condition is not sufficient for quantum states to be separable in general.

For mixed high-dimensional multipartite states, the classification of quantum states becomes much more complicated. It will be helpful to give some discussions that determine if a given state is full multipartite entangled state. Therefore several researches concerning

K. Nagata (✉)
Obihiro University of Agriculture and Veterinary Medicine, Obihiro, Japan
e-mail: ko_mi_na1@yahoo.co.jp

to a sufficient condition of the detection of multipartite entangled state were reported (cf. [18]). However, a necessary condition for a state to be full multipartite entangled state is open question.

In this paper we provide a necessary and sufficient condition for a specific type of states to be full N-qubit entangled state. Here, we shall investigate the property of any N-qubit state which is diagonal in the Greenberger-Horne-Zeilinger (GHZ) [19, 20] basis. It turns out that such a GHZ diagonal state is full N-qubit entangled state if and only if no partial transpose of the multiqubit state is positive with respect to any partition.

Our result is generalization of the result presented in Refs. [7, 8] about a subset of the family of GHZ diagonal states. Dür et al. propose [8] separability criterions about mixed state group $\{\rho_N\}$ in $H=\bigotimes_{j=1}^N C_j^2$. Applying local operations, an arbitrary positive operator ρ of N spin-1/2 systems can be transformed into one of the members of a family of positive operators, i.e.,

$$\rho \to \rho_N = \sum_{\sigma=\pm} \lambda_0^\sigma |\Psi_0^\sigma\rangle\langle\Psi_0^\sigma| + \sum_{h=1}^{2^{(n-1)}-1} \lambda_h (|\Psi_h^+\rangle\langle\Psi_h^+| + |\Psi_h^-\rangle\langle\Psi_h^-|), \tag{2}$$

where $|\Psi_h^\pm\rangle$ represent the orthonormal GHZ basis

$$|\Psi_h^\pm\rangle = \frac{1}{\sqrt{2}}(|h\rangle|0\rangle \pm |2^{N-1}-h-1\rangle|1\rangle), \tag{3}$$

where $h = h_1 h_2 \cdots h_{N-1}$ is understood in binary notation of h. Let us consider three-partite system: Parties A, B, and C.

1. We assume a certain operation is (randomly) performed with $p = 1/2$, while no operation is performed with $p = 1/2$.
2. It is easy to check that the following sequence of mixing operations is sufficient to make ρ diagonal in the GHZ-basis without changing the diagonal coefficients.
3. In the first round we apply simultaneous spin flips at all three locations.
4. In the second and third round we apply σ_z to system B and C or A and C, respectively.
5. Finally, one can depolarize the subspace spanned by $\{|\Psi_h^\pm\rangle\}$ for each $h > 0$ by using random operations that change $|0\rangle \to e^{i\phi_\alpha}|0\rangle_\alpha$ $(\alpha = A, B, C)$ with $\phi_A + \phi_B + \phi_C = 2\pi$ (this condition ensures that $\lambda_0^\pm$ remains unchanged).

We want to omit the final step. The generalization for multi-partite systems from three-partite systems is straightway. Our result is for the general class of GHZ diagonal states.

First we note a definition of full N-qubit entangled state as follows:

Consider a partition of N-qubit system $\mathbf{N}_N = \{1, 2, \ldots, N\}$ into 2 nonempty and disjoint subsets α_1 and α_2, where $|\alpha_1| + |\alpha_2| = N$, to which we refer as a bi-partite split of the system. Let us consider a density operator W on $H = \bigotimes_{j=1}^N H_j$, where H_j represents the subspace (it is a sub-Hilbert space of the full Hilbert space for j) with respect to particle j.

A density operator W may be called bi-separable with respect to a partition α_1, α_2 iff it can be written as

$$W = \sum_l p_l \left(\bigotimes_{i=1}^2 W_l^{\alpha_i} \right), \quad \left(p_l \geq 0, \sum_l p_l = 1 \right), \tag{4}$$

where $W_l^{\alpha_i}, \forall l$ are the density operators on the subspace $\bigotimes_{j\in\alpha_i} H_j$. If a density matrix can be written by a density matrix of the form (4), it is bi-separable with respect to a partition

α_1, α_2. If the density matrix cannot be written by a density matrix of the form (4) with respect to any partition, it is full N-qubit entangled state.

2 Necessary and Sufficient Condition for Bell Diagonal States to Be Entangled

In this section, we review the necessary and sufficient condition for Bell diagonal states to be entangled as follows. Assume Bell diagonal states as

$$\rho = \delta_1|\psi^+\rangle\langle\psi^+| + \delta_2|\psi^-\rangle\langle\psi^-| + \delta_3|\phi^+\rangle\langle\phi^+| + \delta_4|\phi^-\rangle\langle\phi^-| \tag{5}$$

where $A = \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$ and

$$\begin{aligned} |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle + |0,0\rangle), \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle - |0,0\rangle), \\ |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle), \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle). \end{aligned} \tag{6}$$

We know that $\rho^T \geq 0$ is the necessary and sufficient condition for the density matrix ρ to be separable since positivity of partial transpose is the necessary and sufficient condition for quantum states to be separable on $C^2 \otimes C^2$ [16]. The eigenvalues of partial transpose ρ^T are

$$\begin{aligned} B &= \delta_1 - \delta_2 + \delta_3 + \delta_4, \\ C &= \delta_1 + \delta_2 - \delta_3 + \delta_4, \\ D &= \delta_1 + \delta_2 + \delta_3 - \delta_4, \\ E &= -\delta_1 + \delta_2 + \delta_3 + \delta_4. \end{aligned} \tag{7}$$

Thus, if and only if

$$\rho^T \geq 0 \Leftrightarrow B, C, D, E \geq 0, \tag{8}$$

ρ is separable state.

3 Necessary and Sufficient Condition for GHZ Diagonal States to Be Full N-partite Entangled

Consider a subset $\alpha \subset \mathbf{N}_N$ and a density operator W acting on H, let W^{T_α} denote the partial transpose of all sites belonging to α. Let P denote a family of sets, which consists of all unions of α_1, α_2 together with the empty set, so that P has 2^2 elements. A density operator W may be called 2-positive partial transpose (2-PPT) with respect to this specific partition iff $W^{T_\alpha} \geq 0$ for all $\alpha \in P$.

Total transposition preserves the spectrum of a bipartite state. Moreover, total transposition with respect to one side of the partition followed by total transposition corresponds to partial transposition on the other side. Thus, in the partial transposition test, it is sufficient to consider transposition with respect to only the sites belonging to α, (moreover, this means that checking the positivity of the partial transpose with respect to $\alpha = \emptyset$ or $\alpha = \mathbf{N}_N$ would correspond just to check the positivity of the density matrix, which is positive by definition).

Clearly if a density operator W is not 2-PPT with respect to any partition, the state W should be full N-qubit entangled state.

Let β be a subset $\beta \subset \mathbf{N}_N$ and $l(\beta)$ be an integer $l_1 \cdots l_N$ in binary notation with $l_m = 1$ for $m \in \beta$ and $l_m = 0$ otherwise $(0 \le l(\beta) \le 2^N - 1)$.[1]

Using an integer $l(\beta)$ and a subset $\alpha \subset \mathbf{N}_N \wedge \alpha \neq \emptyset$, we introduce two vectors as $\{\bigotimes_{m\in\alpha} |l_m\rangle, \bigotimes_{m\in\alpha} |l_m \oplus 1\rangle\} = \{|B_\beta(\alpha)\rangle, |\overline{B_\beta(\alpha)}\rangle\}$. Here $l_m \oplus 1$ is the bitwise XOR (exclusive OR) of l_m and 1. Hence, $l_m \oplus 1 = 0$ if $l_m = 1$. And $l_m \oplus 1 = 1$ if $l_m = 0$. Here both vectors $|B_\beta(\alpha)\rangle$ and $|\overline{B_\beta(\alpha)}\rangle$ are acting on the subspace $\bigotimes_{m\in\alpha} H_m$. And $\langle B_\beta(\alpha)|\overline{B_\beta(\alpha)}\rangle = 0$ holds for every subset α, β. For a given α, β, two vectors $(|B_\beta(\alpha)\rangle, |\overline{B_\beta(\alpha)}\rangle)$ form a two-dimensional space.

Again, consider a partition of N-qubit system $\mathbf{N}_N = \{1, 2, \dots, N\}$ into 2 nonempty and disjoint subsets α_1 and α_2, where $|\alpha_1| + |\alpha_2| = N$, to which we referred as a bi-partite split of the system.

The orthonormal GHZ basis for 2^N-dimensional space is covered by the following family of states given by

$$|\Psi_\beta^\pm\rangle = \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)|B_\beta(\alpha_2)\rangle \pm |\overline{B_\beta(\alpha_1)}\rangle|\overline{B_\beta(\alpha_2)}\rangle), \quad \beta \subset \mathbf{N}_N. \tag{9}$$

Please notice that

$$|\Psi_\beta^\pm\rangle\langle\Psi_\beta^\pm| = |\Psi_{\mathbf{N}_N\setminus\beta}^\pm\rangle\langle|\Psi_{\mathbf{N}_N\setminus\beta}^\pm|. \tag{10}$$

Therefore, to make the GHZ basis for 2^N-dimensional space, it is sufficient to consider 2^{N-1} kinds of subset (β) even though the cardinal number of subsets of $\mathbf{N}_N$ is 2^N.

In what follows, we shall show that *any N-qubit state which is diagonal in the GHZ basis is full N-qubit entangled state if and only if no partial transpose of the state is positive with respect to any partition.*

Let us consider such a multiqubit density operator X as

$$X = \frac{1}{2}\sum_{\beta\subset\mathbf{N}_N} (\lambda_\beta^+|\Psi_\beta^+\rangle\langle\Psi_\beta^+| + \lambda_\beta^-|\Psi_\beta^-\rangle\langle\Psi_\beta^-|). \tag{11}$$

Of course $(1/2)\sum_{\beta\subset\mathbf{N}_N}(\lambda_\beta^+ + \lambda_\beta^-) = 1,\ \lambda_\beta^+ \ge 0,\ \lambda_\beta^- \ge 0$. The values of the positive coefficients of X, i.e., all λ are

$$\lambda_\beta^\pm = \langle\Psi_\beta^\pm|X|\Psi_\beta^\pm\rangle. \tag{12}$$

In what follows, we shall show that X *is bi-separable with respect to a partition* α_1 *and* α_2 *if and only if the partial transpose with respect to the partition is positive,* $X^{T_{\alpha_1}} \ge 0$.

[1] Therefore by using the following formula $\sum_{i=0}^{N-1} 2^i = 2^N - 1$, we can see that $l(\beta) = \sum_{i=0}^{N-1} l_{N-i}(\beta)2^i$.

We introduce a positive operator as

$$Y_{\beta,\alpha_1,\alpha_2} = \lambda_\beta^+ |\Psi_\beta^+\rangle\langle\Psi_\beta^+| + \lambda_\beta^- |\Psi_\beta^-\rangle\langle\Psi_\beta^-| + \eta_\beta^+ |\Phi_\beta^+\rangle\langle\Phi_\beta^+| + \eta_\beta^- |\Phi_\beta^-\rangle\langle\Phi_\beta^-| \tag{13}$$

where $|\Phi_\beta^\pm\rangle$ is given by

$$|\Phi_\beta^\pm\rangle = \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)\rangle|\overline{B_\beta(\alpha_2)}\rangle \pm |\overline{B_\beta(\alpha_1)}\rangle|B_\beta(\alpha_2)\rangle). \tag{14}$$

Here,

$$\eta_\beta^\pm = \langle\Phi_\beta^\pm|X|\Phi_\beta^\pm\rangle. \tag{15}$$

The operator $Y_{\beta,\alpha_1,\alpha_2}$ is acting on subspace spanned by

$$\begin{aligned}
|\Psi_\beta^+\rangle &= \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)\rangle|B_\beta(\alpha_2)\rangle + |\overline{B_\beta(\alpha_1)}\rangle|\overline{B_\beta(\alpha_2)}\rangle),\\
|\Psi_\beta^-\rangle &= \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)\rangle|B_\beta(\alpha_2)\rangle - |\overline{B_\beta(\alpha_1)}\rangle|\overline{B_\beta(\alpha_2)}\rangle),\\
|\Phi_\beta^+\rangle &= \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)\rangle|\overline{B_\beta(\alpha_2)}\rangle + |\overline{B_\beta(\alpha_1)}\rangle|B_\beta(\alpha_2)\rangle),\\
|\Phi_\beta^-\rangle &= \frac{1}{\sqrt{2}}(|B_\beta(\alpha_1)\rangle|\overline{B_\beta(\alpha_2)}\rangle - |\overline{B_\beta(\alpha_1)}\rangle|B_\beta(\alpha_2)\rangle).
\end{aligned} \tag{16}$$

Clearly, the above set of four vectors is analogous to the set of four vectors presented in (6). Hence, we see that a generic bipartite state is mapped in an operator as in (13). We thus see that an operator in (13) described in N-qubit system is analogous to a quantum state in (5) described in two-qubit system. But normalization factor may be different. That is, $A_\beta = \lambda_\beta^+ + \lambda_\beta^- + \eta_\beta^+ + \eta_\beta^- \le 1$. In order to follow the analogy to two-qubit Bell diagonal states, we introduce the following notations as

$$\begin{aligned}
B_\beta &= \lambda_\beta^+ - \lambda_\beta^- + \eta_\beta^+ + \eta_\beta^-,\\
C_\beta &= \lambda_\beta^+ + \lambda_\beta^- - \eta_\beta^+ + \eta_\beta^-,\\
D_\beta &= \lambda_\beta^+ + \lambda_\beta^- + \eta_\beta^+ - \eta_\beta^-,\\
E_\beta &= -\lambda_\beta^+ + \lambda_\beta^- + \eta_\beta^+ + \eta_\beta^-.
\end{aligned} \tag{17}$$

Then, if and only if

$$Y_{\beta,\alpha_1,\alpha_2}^{T_{\alpha_1}} \ge 0 \quad \Leftrightarrow \quad B_\beta, C_\beta, D_\beta, E_\beta \ge 0. \tag{18}$$

$Y_{\beta,\alpha_1,\alpha_2}/A_\beta$ is bi-separable with respect to a partition α_1 and α_2 since positivity of partial transpose is the necessary and sufficient condition for quantum states to be separable on $C^2 \otimes C^2$ [16]. That is, we have considered the positive operator $Y_{\beta,\alpha_1,\alpha_2}$ as a two-qubit state. Hence, if and only if the condition (18) holds, $Y_{\beta,\alpha_1,\alpha_2}$ can be written as

$$Y_{\beta,\alpha_1,\alpha_2} = \sum_l q_l \left(W_l^{\alpha_1} \otimes W_l^{\alpha_2}\right), \quad \left(q_l \ge 0, \sum_l q_l = A_\beta \le 1\right). \tag{19}$$

One has

$$X = (1/4) \bigotimes_{\beta \subset \mathbf{N}_N} Y_{\beta,\alpha_1,\alpha_2}. \tag{20}$$

Hence, we have

$$X^{T_{\alpha_1}} = (1/4) \bigotimes_{\beta \subset \mathbf{N}_N} Y^{T_{\alpha_1}}_{\beta,\alpha_1,\alpha_2}. \tag{21}$$

This implies $X^{T_{\alpha_1}} \geq 0 \Leftrightarrow Y^{T_{\alpha_1}}_{\beta,\alpha_1,\alpha_2} \geq 0 : \forall \beta \subset \mathbf{N}_N$. Hence, X is bi-separable with respect to a partition α_1 and α_2 if and only if the partial transpose with respect to the partition is positive, $X^{T_{\alpha_1}} \geq 0$.

On the other hand, X is full N-qubit entangled state if there is no partition α_1, α_2 such that $X^{T_{\alpha_1}} \geq 0$.

Our result is constructed by Theorem 3 in [16] and by sub-transformation (not full-transformation) of a quantum state discussed in [8]. We can consider the following example:

$$\begin{aligned} &((1-\epsilon)/2)|GHZ^+\rangle\langle GHZ^+| + ((1-\epsilon)/2)|GHZ^-\rangle\langle GHZ^-| \\ &\quad + \epsilon |GHZ^A\rangle\langle GHZ^A|, \end{aligned} \tag{22}$$

where

$$|GHZ^+\rangle = \frac{|000\ldots0\rangle + |111\ldots1\rangle}{\sqrt{2}}, \tag{23}$$

$$|GHZ^-\rangle = \frac{|000\ldots0\rangle - |111\ldots1\rangle}{\sqrt{2}}, \tag{24}$$

$$|GHZ^A\rangle = \frac{|00\ldots011\ldots1\rangle + |11\ldots100\ldots0\rangle}{\sqrt{2}}. \tag{25}$$

We see that quantum state (22) is full entangled state if $\epsilon > 1/2$.

4 Summary

In summary, we have investigated a specific type of N-qubit states and showed that any N-qubit state which is diagonal in the GHZ basis is bi-separable for a partition if and only if partial transpose of the multiqubit state is positive with respect to the partition. This implies that there is no positive partial transpose for any subsystems if and only if the GHZ diagonal N-qubit state is full N-qubit entangled state.

We cannot generalize our arguments into general multiqubit states, just because there are PPT entangled states of 4 qubits. For instance, in [21], a 4-qubit state is discussed which is positive under transposition on qubits 3, 4, but entangled with respect to the two-component partition $\{1,2,3,4\} = \{1,2\} \cup \{3,4\}$.

Acknowledgements The author thanks G. Tóth and O. Gühne for valuable comments.

References

1. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
2. Galindo, A., Martín-Delgado, M.A.: Rev. Mod. Phys. **74**, 347 (2002)
3. Bell, J.S.: Physics (Long Island City, N.Y.) **1**, 195 (1964)
4. Redhead, M.: Incompleteness, Nonlocality, and Realism, 2nd edn. Clarendon Press, Oxford (1989)
5. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer Academic, Dordrecht (1993)
6. Werner, R.F.: Phys. Rev. A **40**, 4277 (1989)
7. Dür, W., Cirac, J.I., Tarrach, R.: Phys. Rev. Lett. **83**, 3562 (1999)
8. Dür, W., Cirac, J.I.: Phys. Rev. A **61**, 042314 (2000)
9. Kraus, B., Cirac, J.I., Karnas, S., Lewenstein, M.: Phys. Rev. A **61**, 062302 (2000)
10. Karnas, S., Lewenstein, M.: Phys. Rev. A **64**, 042313 (2001)
11. Fei, S.-M., Gao, X.-H., Wang, X.-H., Wang, Z.-X., Wu, K.: Phys. Rev. A **68**, 022315 (2003)
12. Doherty, A.C., Parrilo, P.A., Spedalieri, F.M.: Phys. Rev. A **69**, 022308 (2004)
13. Yu, C.-S., Song, H.-S.: Phys. Rev. A **72**, 022333 (2005)
14. Chen, L., Chen, Y.-X.: Phys. Rev. A **73**, 052310 (2006)
15. Peres, A.: Phys. Rev. Lett. **77**, 1413 (1996)
16. Horodecki, M., Horodecki, P., Horodecki, R.: Phys. Lett. A **223**, 1 (1996)
17. Horodecki, M., Horodecki, P., Horodecki, R.: Phys. Rev. Lett. **80**, 5239 (1998)
18. Tóth, G., Gühne, O.: Phys. Rev. A **72**, 022340 (2005)
19. Greenberger, D.M., Horne, M.A., Zeilinger, A.: In: Kafatos, M. (ed.) Bell's Theorem, Quantum Theory and Conceptions of the Universe, pp. 69–72. Kluwer Academic, Dordrecht (1989)
20. Greenberger, D.M., Horne, M.A., Shimony, A., Zeilinger, A.: Am. J. Phys. **58**, 1131 (1990)
21. Benatti, F., Floreanini, R., Piani, M.: Open Syst. Inf. Dyn. **11**, 325–338 (2004). arXiv:quant-ph/0411095